On the integration of differential equations of higher orders*

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§1 Although many methods have been invented to solve differential equations of first order and the highest Geometers have invested a lot of work and eagerness in this task, nevertheless they have offered hardly anything substantial to treat differential equations of higher orders, either to construct them or to integrate them. Differential equations of second order are usually resolved in such a way that by a suitable substitution they are reduced to equations of first order, having done which their resolution is reduced to a more familiar and known way: And during this task, I invented several auxiliary tools many years ago, by means of which innumerable differential equations of second order can be lowered to first order, and can even be constructed or integrated. But concerning differential equations of third or higher order similar artifices, by which they can be reduced to a lower order, are hardly or even not available, since this way one gets to so complicated differential equations of second or higher order that they can not be treated any further at all. Therefore, the method I am going to explain here and by means of which many differential equations of higher orders can be integrated immediately without any previous reduction and the integral equation can be exhibited in finite terms will be quite useful for this task.

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§2 Let y and x be variables in which the differential equation of arbitrary order is contained, in which equation the element $\mathrm{d}x$ is assumed to be constant, and let the other variable y and its differentials $\mathrm{d}y$, $\mathrm{dd}y$, d^3y etc. have one dimension in each term such that the equation, of whatever order it is, has the following form:

$$0 = Ay + \frac{B\mathrm{d}y}{\mathrm{d}x} + \frac{C\mathrm{dd}y}{\mathrm{d}x^2} + \frac{D\mathrm{d}^3y}{\mathrm{d}x^3} + \frac{E\mathrm{d}^4y}{\mathrm{d}x^4} + \frac{F\mathrm{d}^5y}{\mathrm{d}x^5} + \text{etc},$$

in which the letters A, B, C, D etc. denote either constant quantities or such involving the other variable x in any way. But it is obvious that this equation extends quite far; for, not only because of the general coefficients A, B, C, D etc., which we at the same time assume as arbitrary functions of x, is it very general, but it also contains differential equation of any order as a special case.

§3 First it is certainly perspicuous that, aside from the constant quantities found in the differential equation, the complete integral equation must also contain so many constant quantities as of which order the propounded differential equation was. For, if we put that the order of the differential equation is n, such that the last term of it is

$$\frac{N\mathrm{d}^{\mathrm{n}}y}{\mathrm{d}x^n},$$

by one integration it will be reduced to order $n-1$, by two integrations done successively to order $n-2$, by three to order $n-3$ and so forth. From this it is understood that just after n integrations one gets to an integral equation expressed in finite terms. But since by each integration one arbitrary constant enters the integral, it is obvious that the complete integral must contain n arbitrary constants.

§4 Therefore, the complete integral equation contains as many arbitrary constants as the exponent n contains units; and this integral equation is to be considered to extend as far as the differential equation of order n, such that no finite value assumed for y can satisfy the differential equation which is not also contained in the complete integral equation. But if in this complete integral equation one or more of those arbitrary constants are determined arbitrarily, then one will certainly have an equation answering the question, but it will no longer be complete, but only a particular integral equation, which does not contain all possible values of y satisfying the differential equation. Therefore, the complete integral equation must be distinguished carefully from the particular one; and if we want to satisfy the differential equation perfectly, we have to find the complete integral equation.

§5 But a criterion see, whether the exhibited integral equation is complete or not, is easily derived from the things mentioned. For, first the propounded differential equation must be satisfied, which happens, if after the substitution the identical equation results; for, otherwise that equation would not be the integral equation. Furthermore, it is necessary that the integral equation contains as many arbitrary constants as of which grade the propounded differential equation was. For, if there are less constants in it, then the integral equation will not be complete, but just particular. But in the enumeration of the arbitrary constants one has to be careful not to be fooled by the number of different letters and not to count those as arbitrary constants which depend on each other.

§6 To understand the difference between complete and incomplete differential equations more clearly, it will be helpful to have illustrated this in an example. Therefore, let this differential equation be propounded

$$aa\mathrm{d}y + yy\mathrm{d}x = (aa + xx)\mathrm{d}x,$$

which is clear to be satisfied by the value $y = x$, which substituted in the equation produces the identical equation. Therefore, $y = x$ is an integral equation, but by no means complete, since it neither contains the constant a, which is found in the differential equation, nor another arbitrary constant, as an differential equation of first order requires it. Therefore, somebody would be vehemently wrong, who would want to sell this equation $y = x$ as complete integral of this one

$$aa\mathrm{d}y + yy\mathrm{d}x = (aa + xx)\mathrm{d}x;$$

for, the complete integral is

$$y = x + \frac{aab\mathrm{e}^{-\frac{xx}{aa}}}{aa + b\int \mathrm{e}^{-\frac{xx}{aa}}\mathrm{d}x},$$

which setting the arbitrary constant $b = 0$ gives the particular integral $y = x$, of course.

§7 In like manner, we see that this differential equation of second order

$$y = \frac{x\mathrm{d}y}{\mathrm{d}x} + \frac{ax\mathrm{dd}y}{ax^2}$$

is satisfied by this finite equation $y = x$; but a lot is missing that it is the complete integral and exhausts the complete meaning of the differential equation of second order, since the complete integral equation, aside from the constant a, must contain two arbitrary constants. Indeed, we see that also this equation $y = nx$ satisfies, which, since in contains one single constant n, is still just particular. But the complete integral is

$$y = nx + b\int \frac{e^{-\frac{x}{a}}\mathrm{d}x}{xx},$$

which aside from the constant a contains the two arbitrary constants b and n; as the matter if things require.

§8 But since all particular integral equations are contained in the complete one, it is plain that the complete integral consists of many particular integrals; and hence the complete integral is conflated from particular integrals. Many times it is certainly as difficult to derive the complete integral or at least a further extending integral from several known integral as to derive the same from the differential equation by integration. But the equation we want to treat,

$$0 = Ay + \frac{B\mathrm{d}y}{\mathrm{d}x} + \frac{C\mathrm{dd}y}{\mathrm{d}x^2} + \frac{D\mathrm{d}^3y}{\mathrm{d}x^3} + \frac{E\mathrm{d}^4y}{\mathrm{d}x^4} + \text{etc},$$

is of such a nature that knowing two or more particular values of y from these, one can easily form a further extending value of y containing all those values. And this way one will be able to construct the complete value or the complete integral equation from a sufficient number of particular values.

§9 But first it is understood, if p was a convenient value of y such that $y = p$, that then also will be $y = \alpha p$; and if the value p substituted for y renders

$$Ay + \frac{B\mathrm{d}y}{\mathrm{d}x} + \frac{C\mathrm{dd}y}{\mathrm{d}x^2} + \text{etc} = 0$$

then the value αp substituted for y will cause the same vanishing expression: and this way one arbitrary constant α can be introduced into the particular integral equation $y = p$. But if furthermore $y = q$ satisfies the propounded equation, then in like manner also $y = \beta q$ will satisfy; but from these two particular values $y = \alpha p$ and $y = \beta q$ one will conclude this further extending one

$$y = \alpha p + \beta q.$$

For, if the expression

$$Ay + \frac{B\mathrm{d}y}{\mathrm{d}x} + \frac{C\mathrm{dd}y}{\mathrm{d}x^2} + \frac{D\mathrm{d}^3y}{\mathrm{d}x^3} + \text{etc}$$

is rendered equal to zero having written so αp as for βy for y, it is obvious that at the same expression must become equal to zero, if one writes $\alpha p + \beta y$ instead of y.

§10 In like manner, if p, q, r, s etc. were functions of x of such a kind that each substituted separately for y causes the expression

$$Ay + \frac{B\mathrm{d}y}{\mathrm{d}x} + \frac{C\mathrm{dd}y}{\mathrm{d}x^2} + \text{etc}$$

to vanish, then also this value

$$\alpha p + \beta q + \gamma r + \delta s + \text{etc}$$

if substituted for y will produce an expression equal to zero. Hence if p, q, r, s etc. were particular values of y, which came from the propounded equation, then from them this a lot further extending value is concluded

$$y = \alpha p + \beta q + \gamma r + \delta s + \text{etc}$$

also satisfying the propounded equation. And this value will hence be complete, if so many constants α, β, γ, δ etc. are found as of which order the propounded differential equation was. Therefore, we obtained a simple method to assign the complete value containing all satisfying values of y from many particular values of y: and so one will have the complete integral equation in finite terms.

§11 Therefore, the whole task of finding the complete integral of the propounded differential equation

$$0 = Ay + \frac{B\mathrm{d}y}{\mathrm{d}x} + \frac{C\mathrm{dd}y}{\mathrm{d}x^2} + \frac{D\mathrm{d}^3y}{\mathrm{d}x^3} + \cdots + \frac{N\mathrm{d}^{\mathrm{n}}y}{dx^n}$$

reduces to this that we investigate particular values which substituted for y lead to the identical equation. But so many particular values will be necessary until by collecting them in the prescribed way so many arbitrary constants are there as the highest exponent n contains units. Hence if each particular equation comes with one arbitrary constant, n equations of such a kind are required to constitute the complete integral equation. But if one of these many particular equations contains more than one arbitrary constant, then one will need less particular equations to derive the complete equation from them.

§12 Now let all letters A, B, C, D etc. denote constant quantities such that this differential equation of order n must be integrated

$$0 = Ay + \frac{Bdy}{dx} + \frac{Cddy}{dx^2} + \frac{Dd^3y}{dx^3} + \cdots + \frac{Nd^ny}{dx^n}.$$

Since y with it differentials has the same dimension everywhere, according to my method given in Tomus III. Commentariorum Academiae Pertrpolitanae this differential equation is lowered by one order, if we put

$$y = e^{\int p dx},$$

whence the differentials of y will be

$$\begin{aligned}
\frac{dy}{dx} &= e^{\int p dx} p \\
\frac{ddy}{dx^2} &= e^{\int p dx} \left(pp + \frac{dp}{dx} \right) \\
\frac{d^3y}{dx^3} &= e^{\int p dx} \left(p^3 + \frac{3pdp}{dx} + \frac{ddp}{dx^2} \right) \\
\frac{d^4y}{dx^4} &= e^{\int p dx} \left(p^4 + \frac{6ppdp}{dx} + \frac{4pddp}{dx^2} + \frac{3dp^2}{dx^2} + \frac{d^3p}{dx^3} \right) \\
&\text{etc,}
\end{aligned}$$

if which values are substituted in the propounded equation, it can be divided by $e^{\int p dx}$, and a differential equation of order $n - 1$ will remain.

§13 But here first it is plain, if one takes a constant p, such that its differentials dp, ddp, d^3p etc. vanish, that then, because of the constant A, B, C, D etc., the variable x will go out of the equation completely; and by this hypothesis, the following algebraic equation will result

$$0 = A + Bp + Cp^2 + Dp^3 + Ep^4 + \cdots + Np^n,$$

if from which any value of p is found, one will at the same time have a particular integral $y = e^{px}$ satisfying the propounded differential equation; therefore, as we saw, also this equation $y = \alpha e^{px}$ will satisfy, as often as p was a constant quantity and a root of this algebraic equation

$$0 = A + Bp + Cp^2 + Dp^3 + \cdots + Np^n.$$

§14 Therefore, we reduced the invention of particular values for the variable y to the resolution of an algebraic equation of order n, for which we want to take this one

$$0 = A + Bz + Cz^2 + Dz^3 + \cdots + Nz^n,$$

and each root or divisor of this equation will give as many particular values of y. For, if $py - q$ was a divisor of that equation, from which $z = \frac{p}{q}$ results, it will be

$$y = \alpha \mathrm{e}^{\frac{qx}{p}};$$

this particular value contains one arbitrary constant α. But since that algebraic equation of order n contains n roots or divisors, hence also n particular values will result for y; these taken together will give the universal value for y; and this will at the same time be the complete value, since it contains n arbitrary constants.

§15 Therefore, if all roots of this algebraic equation of order n were real, then the complete value for y will result expressed in real terms, and it will be the aggregate of n exponential formulas of this kind $\alpha e^{qx:p}$, and in this case the integrals can even be expressed only by logarithms or the quadrature of the hyperbola. But if some of the roots of that equation were imaginary, then imaginary exponential formulas will enter into the complete integral; I will teach how to construct these by means of the quadrature of the circle below. The main difficulty occurs, whenever two or more roots of the equation are equal; for, then, because of the equal exponential formulas, the number of arbitrary constants is lowered and for that reason the found integral will not be the complete one anymore.

§16 We will deal with each of both inconveniences, if we contemplate the connection between the propounded differential equation

$$0 = Ay + \frac{Bdy}{dx} + \frac{Cddy}{dx^2} + \frac{Dd^3y}{dx^3} + \cdots + \frac{Nd^ny}{dx^n}$$

and the formed algebraic equation

$$0 = A + Bz + Cz^2 + Dz^3 + \cdots + Nz^n$$

more attentively. For, as the second originates from the first, if one writes z^0 instead of y, z instead of $\frac{dy}{dx}$, and in general z^k instead of $\frac{d^ky}{dx^k}$, so in like manner from each factor of the algebraic equation a differential equation will be formed, which will necessarily be contained in the propounded differential equation and from which hence particular values for y will be found. So, if $pz - q$ or $q - pz$ was a divisor of the algebraic equation, from this by the law of the connection this differential equation originates

$$qy - \frac{pdy}{dx} = 0,$$

which integrated gives

$$y = \alpha e^{\frac{qx}{p}},$$

which is the one which we found from the factor $pz - q$.

§17 Hence it is understood, if one has an arbitrary divisor of that algebraic equation, say $p + qz + rzz$, that then the equation which must arise from this divisor

$$py + \frac{qdy}{dx} + \frac{rddy}{dx^2} = 0$$

gives a value for y which also satisfies the propounded differential equation. From this we will therefore be able to get rid off that difficulty which occurred, if the algebraic equation has two or more equal factors. Therefore, let $(p - qz)^2$ be a divisor of the algebraic equation, and from this in expanded form this differential equation of second order will result

$$ppy - \frac{2pqdy}{dx} + \frac{qqddy}{dx^2} = 0.$$

Let us put

$$y = e^{\frac{px}{q}} u,$$

and after the substitution we will have $ddu = 0$, and hence $u = \alpha + \beta x$. Hence form the quadratic factor $(p - qz)^2$ the following value results

$$y = \mathrm{e}^{\frac{px}{q}}(\alpha + \beta x),$$

which contains two arbitrary constants.

§18 If the algebraic equation has the cubic divisor $(p - qz)^3$, then this equation will be contained in the propounded differential equation

$$p^3 y - \frac{3ppq\mathrm{d}y}{\mathrm{d}x} + \frac{3pqq\mathrm{dd}y}{\mathrm{d}x^2} - \frac{q^3\mathrm{d}^3 y}{\mathrm{d}x^3} = 0,$$

which for

$$y = \mathrm{e}^{\frac{px}{q}} u$$

is transformed into this one: $\mathrm{d}^3 u = 0$; hence $u = \alpha + \beta x + \gamma xx$ originates, from which the propounded equation is satisfied by this value

$$y = \mathrm{e}^{\frac{px}{q}}(\alpha + \beta x + \gamma xx).$$

And if in like manner the algebraic equation

$$0 = A + Bz + Cz^2 + Dz^3 + \cdots + Nz^n$$

has fourth power divisor $(p - qz)^4$, then from it this satisfying particular equation will arise

$$y = \mathrm{e}^{\frac{px}{q}}(\alpha + \beta x + \gamma xx + \delta x^3).$$

And if in general $(p - qz)^k$ is a divisor, the value arising from this will be

$$y = \mathrm{e}^{\frac{px}{q}}(\alpha + \beta x + \gamma xx + \delta x^3 \cdots + \varkappa x^{k-1}),$$

such that it involves k imaginary constants.

§19 Who still has any doubt whether this way from the composite divisors, in which z has more than one dimension, of the equation

$$0 = A + Bz + Cz^2 + Dz^3 + \cdots + Nz^n$$

the values for y are deduced correctly, which satisfy the propounded equation

$$0 = Ay + \frac{B\mathrm{d}y}{\mathrm{d}x} + \frac{C\mathrm{dd}y}{\mathrm{d}x^2} + \frac{D\mathrm{d}^3 y}{\mathrm{d}x^3} + \cdots + \frac{N\mathrm{d}^n y}{\mathrm{d}x^n},$$

this doubt will be removed easily by the matter of things. Let a somehow composite divisor be

$$p + qz + rzz + sz^3 + \text{etc}$$

and from it form the equation

$$0 = py + \frac{q\mathrm{d}y}{\mathrm{d}x} + \frac{r\mathrm{dd}y}{\mathrm{d}y^2} + \frac{s\mathrm{d}^3y}{\mathrm{d}x^3} + \text{etc};$$

and it will become clear that complete value of y for this equation results, of all values of y which the simple divisors of this equation give

$$0 = p + qz + rzz + sz^3 + \text{etc},$$

are collected into one sum; but the simple divisors of this equation are at the same time the simple divisors of this one

$$0 = A + Bz + Cz^2 + Dz^3 + \cdots + Nz^n;$$

and for this reason the value of y arising from that composite factor it at the same time a satisfying value of the propounded differential equation

$$0 = Ay + \frac{B\mathrm{d}y}{\mathrm{d}x} + \frac{C\mathrm{dd}y}{\mathrm{d}x^2} + \frac{D\mathrm{d}^3y}{\mathrm{d}x^3} + \cdots + \frac{N\mathrm{d}^{\mathrm{n}}y}{\mathrm{d}x^n}.$$

§20 But having found the values of y resulting from several equal simple divisors of the equation

$$0 = A + Bz + Czz + Dz^3 + \cdots + Nz^n,$$

the other difficulty we have to solve remains, i.e. if this equation has imaginary roots. But it is known, if a certain equation has imaginary roots, that their number will always be even; and I showed on another occasion that by conjugating these imaginary roots they can be split into pairs of such a kind the sum and product of which become real. Hence instead of imaginary divisors real composite divisors of degree two of this form will result

$$p - qz + rzz,$$

which have imaginary simple divisors. Therefore, in such a composite divisor $qq < 4pr$; hence

$$\frac{q}{2\sqrt{pr}} < 1.$$

Therefore, having put the complete sine $= 1$ $\frac{q}{2\sqrt{pq}}$, will be the cosine of a real angle, which shall be $= \varphi$ and it will be[1]

$$q = 2\sqrt{pr}\cos A\varphi,$$

from which the general form of the composite divisors the imaginary factors contain will be of this kind

$$p - 2z\sqrt{pr}\cos A\varphi + rzz.$$

§21 Therefore, let

$$p - 2z\sqrt{pr}\cos A\varphi + rzz$$

be a divisor of such a kind of the equation

$$0 = A + Bz + Cz^2 + \text{etc},$$

from which the corresponding value of y must be found. But from this divisor this differential equation of second order results

$$0 = py - \frac{2\mathrm{d}y\sqrt{pr}}{\mathrm{d}x}\cos A\varphi + \frac{r\mathrm{dd}y}{\mathrm{d}x^2},$$

to integrate which put

$$y = \mathrm{e}^{fx\cos A\varphi}u$$

and, for the sake of brevity having set $f = \sqrt{\frac{p}{r}}$, it will be

$$ffu\mathrm{d}x^2\sin^2 A\varphi + \mathrm{dd}u = 0.$$

Multiply by $2du$ and integrate, it will be

$$ffuu\mathrm{d}x^2\sin^2 A\varphi + \mathrm{d}u^2 = \alpha^2 ff\mathrm{d}x^2\sin^2 A\varphi,$$

whence

$$f\mathrm{d}x\sin A\varphi = \frac{\mathrm{d}u}{\sqrt{\alpha^2 - u^2}};$$

this gives integrated

$$fx\sin A\varphi + \beta = A\sin\frac{u}{\alpha}.$$

[1]Note: In the following A is not to be understood as a constant, instead Euler writes $\cos A\varphi$ and $\sin A\varphi$ instead of the modern $\cos\varphi$ and $\sin\varphi$ in this paper. A stands for the Latin word *Arcus* i.e. arc in English.

From this equation
$$u = \alpha \sin A(fx \sin A\varphi + \beta).$$
As a logical consequence one has
$$y = \alpha e^{fx \cos A\varphi} \sin A(fx \sin A\varphi + \beta),$$
which will be the corresponding value of y for the propounded equation.

§22 The same or an equivalent expression for y is derived from the simple even though imaginary factors of the equation
$$0 = p - 2z\sqrt{pr}\cos A\varphi + rzz,$$
which for $f = \sqrt{\frac{p}{r}}$ goes over into this one
$$0 = ff - 2fz \cos A\varphi + zz,$$
the roots of which are
$$z = f \cos A\varphi \pm f\sqrt{-1}\sin A\varphi.$$
Hence for y these values result
$$e^{fx\cos A\varphi + fx\sqrt{-1}\sin A\varphi} \quad \text{and} \quad e^{fx\cos A\varphi - fx\sqrt{-1}\sin A\varphi},$$
having combined which
$$y = e^{fx\cos A\varphi}\left(\eta e^{+fx\sqrt{-1}\sin A\varphi} + \theta e^{-fx\sqrt{-1}\sin A\varphi}\right).$$
But having converted these exponentials into series it will result
$$y = e^{fx\cos A\varphi}\begin{cases}(\eta+\theta)\left(1 - \frac{ffxx\sin^2 A\varphi}{1\cdot 2} + \frac{f^4x^4\sin^4 A\varphi}{1\cdot 2\cdot 3\cdot 4} + \text{etc}\right) \\ (\eta-\theta)\sqrt{-1}\left(fx\sin A\varphi - \frac{f^3x^3\sin^3 A\varphi}{1\cdot 2\cdot 3} + \text{etc}\right).\end{cases}$$
Therefore, having put
$$\eta = \theta + \alpha \quad \text{and} \quad (\eta - \theta)\sqrt{-1} = \beta$$
and having summed these infinite series it will result
$$y = e^{fx\cos A\varphi}(\alpha \cos A\varphi fx \sin A\varphi + \beta \sin Afx\sin A\varphi);$$
this expression is easily reduced to the first.

§23 Hence we obtain a method to find the value of y, if two or more of these composite divisors were equal. For, let

$$(ff-2fz\cos A\varphi+zz)^2$$

be a divisor of the algebraic equation; since it is reduced to

$$(z-f\cos A\varphi-f\sqrt{-1}\sin A\varphi)^2(z-f\cos A\varphi+f\sqrt{-1}\sin A\varphi)^2,$$

by the preceding results the value which must originate from this for y will be

$$y=e^{fx\cos A\varphi+fx\sqrt{-1}\sin A\varphi})(\eta+\theta x)+e^{fx\cos A\varphi-fx\sqrt{-1}\sin A\varphi}(\iota+\varkappa x).$$

But since

$$\begin{aligned}&e^{+fx\sqrt{-1}\sin A\varphi}\eta+e^{-fx\sqrt{-1}\sin A\varphi}\iota\\=&\alpha\cos Afx\sin A\varphi+\beta\sin Afx\sin A\varphi,\end{aligned}$$

one hence concludes that it will be

$$y=e^{fx\cos A\varphi}\left[(\alpha+\beta x)\cos Afx\sin A\varphi+(\gamma+\delta x)\sin Afx\sin A\varphi\right].$$

§24 But if a cube or another power of

$$ff-2fz\cos A\varphi+zz$$

was a divisor of the algebraic equation

$$0=A+Bz+Czz+Dz^3+\cdots+Nz^n,$$

then from the same powers of simple imaginary factors find the values of y according to § 18 and combine them into one sum. Having done this the imaginary exponential quantities can be converted into sines and cosines of circular arcs by means of this lemma

$$\begin{aligned}&e^{+fx\sqrt{-1}\sin A\varphi}\eta x^k+e^{-fx\sqrt{-1}\sin A\varphi}\theta x^k\\=&\alpha x^k\cos Afx\sin A\varphi+\beta x^k\sin Afx\sin A\varphi.\end{aligned}$$

So if

$$(ff-2fz\cos A\varphi+zz)^4$$

was a divisor of the algebraic equation, then from it the following integral equation will originate

$$\begin{aligned}y=e^{fx\cos A\varphi}&\left[(\alpha+\beta x+\gamma x^2+\delta x^3)\cos Afx\sin A\varphi\right.\\&\left.+(\varepsilon+\zeta x+\eta x^2+\theta x^3)\sin Afx\sin A\varphi\right].\end{aligned}$$

§25 Those expressions can be transformed in many ways, depending on whether the constants are expressed in the one or the other way. But that transformation seems to be most convenient in which the values of y are reduced to the for found in § 21. So this form

$$\mu x^k \cos Afx \sin A\varphi + \nu x^k \sin Afx \sin A\varphi,$$

if one puts

$$\mu = \lambda \sin Ap \quad \text{and} \quad \nu = \lambda \cos Ap,$$

will be transformed into this one

$$\lambda x^k \sin A(fx \sin A\varphi + p).$$

For that reason from the indefinite factor of the exponent

$$(ff - 2fz \cos A\varphi + zz)^k$$

the following value of y will be formed:

$$y = e^{fx \cos A\varphi}(\alpha \sin A(fx \sin A\varphi + \mathfrak{A}) + \beta x \sin A(fx \sin A\varphi + \mathfrak{B})$$
$$+\gamma x^2 \sin A(fx \sin A\varphi + \mathfrak{C}) + \cdots + \varkappa x^{k-1} \sin A(fx \sin A\varphi + \mathfrak{K})),$$

and this way from all divisors, however they were composited, real values for the variable y are found.

§26 Concerning the arbitrary constants entering into the values of y to be found this way, it is plain that first from the simple factors of the form $f - z$ values of y containing one arbitrary constant result; further, the value of y originating from the factor $(f - z)^k$ contains k arbitrary constants. Furthermore, from the composite factor

$$ff - 2fz \cos A\varphi + zz$$

a value of y containing two arbitrary constants results; and from a power of factors of this kind

$$(ff - 2fz \cos A\varphi + zz)^k$$

a value of y is formed, in which $2k$ arbitrary constants are found; such that the number of arbitrary constants is equal to the number of dimensions of z this variable obtains in the divisor, from which the values of y is found.

§27 Therefore, if the algebraic equation we formed from the propounded differential equation

$$0 = A + Bz + Cz^2 + Dz^3 + Ez^4 + \cdots + Nz^n,$$

is resolved into its either simple or composite real factors or such which are powers of the simple or composite ones, and in the described way from each one of them the corresponding values of y are formed, then all these values of y considered together will contain so many arbitrary constants as units are found in the exponent n. Therefore, all these values collected into one sum will not only yield a value for y satisfying the propounded equation

$$0 = Ay + \frac{Bdy}{dx} + \frac{Cddy}{dx^2} + \frac{Dd^3y}{dx^3} + \cdots + \frac{Nd^ny}{dx^n},$$

but this will be the complete value of y containing all possible values satisfying this equation. Therefore, this way that differential equation is integrated perfectly in finite terms, and the integral never requires any other quadratures than those of the hyperbola and the circle.

Problem I

If a differential equation of order n of this kind was propounded

$$0 = Ay + \frac{Bdy}{dx} + \frac{Cddy}{dx^2} + \frac{Dd^3y}{dx^3} + \cdots + \frac{Nd^ny}{dx^n},$$

in which the element dx is put constant, and the letters $A, B, C, D, \cdots, N$ denote arbitrary constant coefficients, to find the integral of this equation in finite real terms.

Solution

Write 1 instead of y, z instead of $\frac{dy}{dx}$, z^2 instead of $\frac{ddy}{dx^2}$ and in general z^k instead of $\frac{d^ky}{dx^k}$; and hence form the following algebraic equation of n dimensions

$$0 = A + Bz + Cz^2 + Dz^3 + \cdots + Nz^n.$$

Further, find all real simple divisors of this equation; and, if it has imaginary divisors, instead of them take composite real divisors, in which z has two dimensions, since a two imaginary factors always constitute one composite real factor. Furthermore, from each divisor form particular values of y in the following way. Of course, from each simple factor, if there are no others equal to it, of this form $f - z$ this value originates

$$y = \alpha e^{fx}.$$

But from two or more equal factors taken together the values of y must be determined. For, from the factor $(f - z)^2$ this value originates

$$y = (\alpha + \beta x)e^{fx},$$

from the factor $(f - z)^3$ this value originates

$$y = (\alpha + \beta x + \gamma xx)e^{fx};$$

and in general from the factor $(f - z)^k$ one deduces

$$y = e^{fx}(\alpha + \beta x + \gamma xx + \cdots + \varkappa x^{k-1}).$$

Concerning composite factors, if that algebraic equation has the factor

$$ff - 2fz\cos A\varphi + zz,$$

which is not a multiple factor, the value which must arise from it will be

$$y = e^{fx\cos A\varphi}\alpha \sin A(fx\sin A\varphi + \mathfrak{A}).$$

If the algebraic equation has two equal factors of this kind such that it is divisible by

$$(ff - 2fz\cos A\varphi + zz)^2.$$

then from this quadratic divisor the following value results

$$y = \alpha e^{fx\cos A\varphi}\sin A(fx\sin A\varphi + \mathfrak{A}) + \beta x e^{fx\cos A\varphi}\sin A(fx\sin A\varphi + \mathfrak{B}).$$

But if any arbitrary power of this factor, say

$$(ff - 2fz\cos A\varphi + zz)^k,$$

was a divisor of the algebraic equation, then from it the following value results

$$\begin{aligned} y = {} & \alpha e^{fx\cos A\varphi}\sin A(fx\sin A\varphi + \mathfrak{A}) + \beta x e^{fx\cos A\varphi}\sin A(fx\sin A\varphi + \mathfrak{B}) \\ & + \gamma x^2 e^{fx\cos A\varphi}\sin A(fx\sin A\varphi + \mathfrak{C}) + \delta x^3 e^{fx\cos A\varphi}\sin A(fx\sin A\varphi + \mathfrak{D}) \\ & + \cdots + \varkappa x^{k-1}e^{fx\cos A\varphi}\sin A(fx\sin A\varphi + \mathfrak{K}). \end{aligned}$$

But having found the respective values of y from each divisor of the algebraic equation this way, it just remains that all these values are collected into one sum, and this way the complete value of y will result; and this is the one which would have resulted, if the propounded differential equation of order n would be integrated n times. Q.E.I.

EXAMPLE 1

§29 To find the integral of this differential equation of second order

$$0 = ay + \frac{bdy}{dx} + \frac{cddy}{dx^2}.$$

Having written 1 instead of y, z instead of $\frac{dy}{dx}$ and zz instead of $\frac{ddy}{dx^2}$, as it was prescribed, this equation results

$$0 = a + bz + czz;$$

it will either have two real roots or two imaginary ones; the first happens, if $bb > 4ac$, the second, if $bb < 4ac$. Therefore, let $bb > 4ac$ first, and the two roots will be

$$z = \frac{-b \pm \sqrt{bb - 4ac}}{2c}$$

and in this case the integral in question will be

$$y = \alpha e^{\frac{-bx + x\sqrt{bb-4ac}}{2c}} + \beta e^{\frac{-bx - x\sqrt{bb-4ac}}{2c}}.$$

Here the case $bb = 4ac$ is to be considered separately, for, then

$$a + 2z\sqrt{ac} + czz$$

will be a square, namely

$$(\sqrt{a} + z\sqrt{c})^2,$$

which compared to the form $(f - z)^2$ gives

$$f = -\sqrt{\frac{a}{c}},$$

whence the integral will be

$$y = (\alpha + \beta x)e^{-x\sqrt{\frac{a}{c}}},$$

which is the integral of the equation

$$0 = ay + \frac{2\mathrm{d}y\sqrt{ac}}{\mathrm{d}x} + \frac{c\mathrm{dd}y}{\mathrm{d}x^2}.$$

Now let $bb < 4ac$, and the equation

$$0 = a + bz + czz$$

will have no real roots, therefore, compared to the formula

$$ff - 2fz\cos A\varphi + zz$$

it gives

$$\frac{b}{c} = -2f\cos A\varphi \quad \text{and} \quad \frac{a}{c} = ff;$$

hence it will be

$$f = \sqrt{\frac{a}{c}} \quad \text{and} \quad \cos A\varphi = \frac{-b}{2\sqrt{ac}}$$

and thus

$$\sin A\varphi = \frac{\sqrt{4ac - bb}}{2\sqrt{ac}},$$

whence the following integral results

$$y = \alpha e^{\frac{-bx}{2c}} \sin A\left(\frac{x\sqrt{4ac-bb}}{2c} + \mathfrak{A}\right).$$

EXAMPLE 2

§30 To find the integral of this differential equation of third order

$$0 = y - \frac{3a^2\mathrm{dd}y}{\mathrm{d}x^2} + \frac{2a^3\mathrm{d}^3y}{\mathrm{d}x^3}.$$

Therefore, from that equation this algebraic one results

$$0 = 1 - 3a^2zz + 2a^3z^3,$$

which is resolved into these factors

$$(1 + 2az), \quad (1 - az)^2.$$

The first factor $1 + 2az$ compared to $f - z$ gives

$$f = \frac{-1}{2a},$$

whence it results

$$y = \alpha \mathrm{e}^{-\frac{x}{2\alpha}};$$

the second factor $(1 - az)^2$ must be compared to $(f - z)^2$, from which

$$f = \frac{1}{a}$$

and hence it results

$$y = (\beta + \gamma x)\mathrm{e}^{\frac{x}{a}}.$$

Therefore, the complete integral of the propounded equation will be

$$y = \alpha \mathrm{e}^{-\frac{x}{2a}} + (\beta + \gamma x)\mathrm{e}^{\frac{x}{a}}.$$

EXAMPLE 3

§31 To find the integral of this differential equation of third order

$$0 = y - \frac{a^3 \mathrm{d}^3 y}{\mathrm{d}x^3}.$$

The algebraic equation resulting from this equation will be

$$0 = 1 - a^3 z^3,$$

which is resolved into these factors:

$$(1 - az), \quad (1 + az + a^2 zz),$$

such that its divisors are

$$\frac{1}{a} - z \quad \text{and} \quad \frac{1}{aa} + \frac{z}{a} + zz,$$

the second of which can not be resolved into real simple ones. Therefore, that divisor $\frac{1}{a} - z$ gives for the integral

$$y = \alpha \mathrm{e}^{\frac{x}{a}},$$

the other divisor

$$\frac{1}{aa}+\frac{z}{a}+zz$$

compared to the form

$$ff-2fz\cos A\varphi+zz$$

gives

$$f=\frac{1}{a}\quad\text{and}\quad\frac{-2\cos A\varphi}{a}=\frac{1}{a},$$

such that

$$\cos A\varphi=-\frac{1}{2}\quad\text{and}\quad\sin A\varphi=\frac{\sqrt{3}}{2};$$

hence from this divisor it results

$$y=\beta e^{-\frac{x}{2a}}\sin A\left(\frac{x\sqrt{3}}{2a}+\mathfrak{A}\right).$$

Therefore, the complete integral of the propounded equation will be

$$y=\alpha e^{\frac{x}{a}}+\beta e^{-\frac{x}{2a}}\sin A\left(\frac{x\sqrt{3}}{2a}+\mathfrak{A}\right).$$

EXAMPLE 4

§32 To find the integral of this differential equation of fourth order

$$0=y-\frac{a^4d^4y}{dx^4}.$$

From this equation this algebraic equation will be formed

$$0=1-a^4z^4,$$

which has to simple real divisors

$$\frac{1}{a}-z\quad\text{and}\quad\frac{1}{a}+z;$$

the remaining two imaginary ones are contained in this composite one

$$\frac{1}{aa}+zz.$$

The two simple divisors give for the integral

$$y = \alpha x^{\frac{x}{a}} + \beta \mathrm{e}^{-\frac{x}{a}}.$$

But the divisor

$$\frac{1}{aa} + zz$$

compared to the form

$$ff - 2fz\cos A\varphi + zz$$

gives

$$f = \frac{1}{a} \quad \text{and} \quad \cos A\varphi = 0$$

and hence

$$\sin A\varphi = 1.$$

Therefore, the exponential term

$$\mathrm{e}^{fx\cos A\varphi},$$

because of the exponent $= 0$, becomes 1, and it will be

$$y = \gamma \sin A\left(\frac{x}{a} + \mathfrak{A}\right).$$

Therefore, the complete integral will be

$$y = \alpha \mathrm{e}^{\frac{x}{a}} + \beta \mathrm{e}^{-\frac{x}{a}} + \gamma \sin A\left(\frac{x}{a} + \mathfrak{A}\right).$$

EXAMPLE 5

§33 To find the integral of this differential equation of fourth order

$$0 = y + \frac{a^4 \mathrm{d}^4 y}{\mathrm{d}x^4}.$$

Therefore, one will have to resolve this algebraic equation

$$0 = 1 + a^4 z^4;$$

since it has no simple real divisor, it is resolved into these two composite real factors

$$1 + az\sqrt{2} + aazz \quad \text{and} \quad 1 - az\sqrt{2} + aazz,$$

which divided by aa, that they can be compared to the form

$$ff - 2fz\cos A\varphi + zz,$$

will give

$$\frac{1}{aa} + \frac{z\sqrt{2}}{a} + zz \quad \text{and} \quad \frac{1}{aa} - \frac{z\sqrt{2}}{a} + zz;$$

for each of them $f = \frac{1}{a}$; but for the first

$$f\cos A\varphi = \frac{-1}{a\sqrt{2}},$$

for the second

$$f\cos A\varphi = \frac{1}{a\sqrt{2}};$$

and hence for each of them

$$f\sin A\varphi = \frac{1}{a\sqrt{2}}.$$

From these the complete integral of the propounded equation results

$$y = \alpha e^{-\frac{x}{a\sqrt{2}}}\sin A\left(\frac{x}{a\sqrt{2}} + \mathfrak{A}\right) + \beta e^{\frac{x}{a\sqrt{2}}}\sin A\left(\frac{x}{a\sqrt{2}} + \mathfrak{B}\right).$$

EXAMPLE 6

§34 To find the complete integral of this differential equation of seventh order

$$0 = y + \frac{ddy}{dx^2} + \frac{d^3y}{dx^3} + \frac{d^4y}{dx^4} + \frac{d^5y}{dx^5} + \frac{d^7y}{dx^7}$$

Hence this algebraic equation of seventh order results

$$0 = 1 + zz + z^3 + z^4 + z^5 + z^7,$$

which is resolved into the following real simple and composite factors

$$(1+z), \quad (1+z+zz), \quad (1-z+zz)^2.$$

The first of them compared to the form $f - z$ gives $f = -1$, and hence it results

$$y = \alpha e^{-x}.$$

But the factor $1+z+zz$ compared to

$$ff - 2fz\cos A\varphi + zz$$

gives

$$f = 1 \quad \text{and} \quad \cos A\varphi = -\frac{1}{2},$$

whence

$$\sin A\varphi = \frac{\sqrt{3}}{2}$$

and the integral arising from this

$$y = \beta e^{-\frac{x}{2}} \sin A\left(\frac{x\sqrt{3}}{2} + \mathfrak{A}\right).$$

The third factor $(1-z+zz)^2$ must be compared to

$$(ff - 2fz\cos A\varphi + zz)^2,$$

whence

$$f = 1, \quad \cos A\varphi = \frac{1}{2} \quad \text{and} \quad \sin A\varphi = \frac{\sqrt{3}}{2}.$$

Therefore, hence this integral results

$$y = \gamma e^{\frac{x}{2}} \sin A\left(\frac{x\sqrt{3}}{2} + \mathfrak{B}\right) + \delta x e^{\frac{x}{2}} \sin A\left(\frac{x\sqrt{3}}{2} + \mathfrak{C}\right).$$

For that reason the complete integral of the propounded differential equation will be

$$\begin{aligned} y = &\alpha e^{-x} + \beta e^{-\frac{x}{2}} \sin A\left(\frac{x\sqrt{3}}{2} + \mathfrak{A}\right) \\ &+ \gamma e^{\frac{x}{2}} \sin A\left(\frac{x\sqrt{3}}{2} + \mathfrak{B}\right) + \delta x e^{\frac{x}{2}} \left(\frac{x\sqrt{3}}{2} + \mathfrak{C}\right), \end{aligned}$$

in which seven arbitrary constants are contained, of course.

EXAMPLE 7

§35 To find the complete integral of this differential equation of eighth order

$$0 = \frac{d^3y}{dx^3} - \frac{3d^4y}{dx^4} + \frac{4d^5y}{dx^5} - \frac{4d^6y}{dx^6} + \frac{3d^7y}{dx^7} - \frac{d^8y}{dx^8}.$$

The algebraic equation of eighth order that must be resolved will be

$$0 = z^3 - 3z^4 + 4z^5 - 4z^6 + 3z^7 - z^8;$$

this is at first clear to be divisible by z^3, which divisor compared to the form $(f - z)^3$ gives $f = 0$, and hence one finds for the integral

$$y = \alpha + \beta x + \gamma xx.$$

Having taken this divisor into account it remains to resolve this equation

$$0 = 1 - 3z + 4zz - 4z^3 + 3z^4 - z^5,$$

which is detected to be divisible by $1 + zz$, having compared which to the form

$$ff - 2fz\cos A\varphi + zz$$

we find

$$f = 1 \quad \text{and} \quad \cos A\varphi = 0,$$

whence $\sin A\varphi = 1$; and hence it results

$$y = \delta \sin A(x + \mathfrak{A}).$$

Further, having done the division by $1 + zz$, this equation remains

$$1 - 3z + 3zz - z^3 = 0 = (1 - z)^3;$$

therefore, in the form $(f - z)^3$ we have $f = 1$, and the integral which must result from this is

$$y = (\varepsilon + \zeta x + \eta xx)e^x.$$

As a logical consequence the complete integral of the propounded equation is

$$y = \alpha + \beta x + \gamma xx + \delta \sin A(x + \mathfrak{A}) + (\varepsilon + \zeta x + \eta xx)e^x.$$

EXAMPLE 8

§36 To find the integral of this differential equation of indefinite order

$$0 = \frac{d^n y}{dx^n}.$$

Hence this algebraic equation results

$$z^n = 0;$$

since all its roots are equal, it must be compared to the factor $(f - z)^k$, and it will be $k = n$ and $f = 0$, from which immediately the integral in question results as

$$y = \alpha + \beta x + \gamma x^2 + \delta x^3 + \cdots + \nu x^{n-1}.$$

But this same integral is easily found by n successive integrations. For, in the first integration it results

$$\alpha = \frac{d^{n-1} y}{dx^{n-1}};$$

multiply by dx and integrate a second time, it will be

$$\alpha x + \beta = \frac{d^{n-2} y}{dx^{n-2}}.$$

This one multiplied by dx and integrated a third time will give

$$\frac{\alpha xx}{2} + \beta x + \gamma = \frac{d^{n-3} y}{dx^{n-3}}$$

and so forth, if the integration is repeated n times, having changed the expressions of the constants the integral will result which we found by our rule.

§37 By means of this method one can also integrate many other differential equations of indefinite degree, which lead to algebraic equations, the real either simple or composite factors of which can actually be exhibited. But since this is not the place to give a method to investigate divisors of equations of an indefinite number of dimensions of this kind, here we will additionally treat differential equations of such a kind which lead to algebraic equations the factors of which are already known from elsewhere. But equations of this kind are

$$f^n \pm z^n = 0 \quad \text{and} \quad f^{2n} \pm 2pf^n z^n \pm z^{2n} = 0;$$

for, all real so simple as composite or trinomial factors of these expressions were exhibited by the supreme mathematicians Cotes and de Moivre, which we will therefore assume as known in the solutions of the following problems.

Problem II

§38 If this differential equation of order n was propounded

$$0 = y - \frac{\mathrm{d}^n y}{\mathrm{d}x^n},$$

in which the element $\mathrm{d}x$ is put constant, to find its complete integral.

Solution

Haven written 1 instead of y and z^n instead of $\frac{\mathrm{d}^n y}{\mathrm{d}x^n}$, as we prescribed, one will have this algebraic equation

$$0 = 1 - z^n,$$

of which $1+z$ is always a simple divisor and, if n was an even number, also $1+z$ is a simple divisor. But all remaining simple divisors are imaginary, and they are contained in this general form

$$1 - 2z\cos A\frac{2k\pi}{n} + zz$$

(where π denotes half of the circumference of the circle whose radius is $=1$); this compared to the general trinomial factor

$$ff - 2fz\cos A\varphi + zz$$

gives

$$f = 1 \quad \text{and} \quad \varphi = \frac{2k\pi}{n},$$

such that this divisor gives the integral value

$$y = \alpha e^{x\cos A\frac{2k\pi}{n}}\sin A\left(x\sin A\frac{2k\pi}{n} + \mathfrak{A}\right).$$

If now all even numbers not exceeding the exponent n are successively substituted for $2k$, all possible values will result, which substituted for y satisfy the equation. Indeed, even the value of y arsing from the simple factor $1-z$, which is $y=\alpha e^{x}$, is contained in this general form; for, having put $k=0$

$$\cos A\frac{2k\pi}{n}=1 \quad \text{and} \quad \sin A\frac{2k\pi}{n}=0$$

and hence $y=\alpha e^{x}$, because of the constant $\sin A\mathfrak{A}$ absorbed into the constant. In like manner, if n is an even number, the value of y which must arise from the factor $1+z$, which is $y=\alpha e^{-x}$, results from the general factor for $2k=n$; for, then

$$\cos A\frac{2k\pi}{n}=-1 \quad \text{and} \quad \sin A\frac{2k\pi}{n}=0,$$

such that the value which must result from the general factor is $y=\alpha e^{-x}$. Therefore, the complete integral will be obtained, if in this general form

$$y=\alpha \mathrm{e}^{x\cos A\frac{2k\pi}{n}}\sin A\left(x\sin A\frac{2k\pi}{n}+\mathfrak{A}\right)$$

all even numbers from 0 to n are successively substituted and these values are collected into one sum. Therefore, the complete integral in question will result

$$\begin{aligned} y=\alpha \mathrm{e}^{x}&+\beta \mathrm{e}^{x\cos A\frac{2\pi}{n}}\sin A\left(x\sin A\frac{2\pi}{n}+\mathfrak{B}\right)\\ &+\gamma \mathrm{e}^{x\cos A\frac{4\pi}{n}}\sin A\left(x\sin A\frac{4\pi}{n}+\mathfrak{C}\right)\\ &+\delta \mathrm{e}^{x\cos A\frac{6\pi}{n}}\sin A\left(x\sin A\frac{6\pi}{n}+\mathfrak{D}\right)\\ &+\varepsilon \mathrm{e}^{x\cos A\frac{8\pi}{n}}\sin A\left(x\sin A\frac{8\pi}{n}+\mathfrak{E}\right)\\ &+\text{etc},\end{aligned}$$

which terms must be continued until one has n arbitrary constants, or, what is the same, until the coefficient of π is larger than 1. But, if n was an odd number, the last term will be

$$=\nu \mathrm{e}^{x\cos A\frac{(n-1)}{n}\pi}\sin A\left(x\sin\frac{(n-1)\pi}{n}+\mathfrak{N}\right);$$

but if n is an even number, the last term will be $=\nu e^{-x}$ and the penultimate term will be

$$=\mu \mathrm{e}^{x\cos A\frac{(n-2)}{n}\pi}\sin A\left(x\sin A\frac{(n-2)\pi}{n}+\mathfrak{M}\right).$$

Therefore, for each value of n the complete integral will be assigned in convenient manner. Q.E.I.

§39 To present these integrals more clearly, let us exhibit the integral of the equation

$$0 = y - \frac{\mathrm{d}^n y}{\mathrm{d}x^n}$$

for each value of n starting from 1:

1. The integral of this equation $0 = y - \frac{\mathrm{d}y}{\mathrm{d}x}$

$$y = \alpha \mathrm{e}^x.$$

2. The integral of this equation $0 = y - \frac{\mathrm{dd}y}{\mathrm{d}x^2}$ is:

$$y = \alpha \mathrm{e}^x + \beta \mathrm{e}^{-x}.$$

3. The integral of this integral $0 = y - \frac{\mathrm{d}^3 y}{\mathrm{d}x^3}$ is:

$$y = \alpha \mathrm{e}^x + \beta \mathrm{e}^{x \cos A \frac{2}{3}\pi} \sin A \left(x \sin A \frac{2}{3}\pi + \mathfrak{B} \right).$$

4. The integral of this equation is $0 = y - \frac{\mathrm{d}^4 y}{\mathrm{d}x^4}$ is:

$$y = \alpha \mathrm{e}^x + \beta \sin A \left(x + \mathfrak{B} \right) + \gamma \mathrm{e}^{-x}.$$

5. The integral of this equation $0 = y - \frac{\mathrm{d}^5 y}{\mathrm{d}x^5}$ is:

$$y = \alpha \mathrm{e}^x + \beta \mathrm{e}^{x \cos A \frac{2}{5}\pi} \sin A \left(x \sin A \frac{2}{5}\pi + \mathfrak{B} \right) + \gamma \mathrm{e}^{x \cos A \frac{4}{5}\pi} \sin A \left(x \sin A \frac{4}{5}\pi + \mathfrak{C} \right).$$

6. The integral of this equation $0 = y - \frac{\mathrm{d}^6 y}{\mathrm{d}x^6}$ is:

$$y = \alpha \mathrm{e}^x + \beta \mathrm{e}^{x \cos A \frac{1}{3}\pi} \sin A \left(x \sin A \frac{1}{3}\pi + \mathfrak{B} \right) + \gamma \mathrm{e}^{x \cos A \frac{2}{3}\pi} \sin A \left(x \sin A \frac{2}{3}\pi + \mathfrak{C} \right) + \delta \mathrm{e}^{-x}.$$

7. The integral of this equation $0 = y - \frac{d^7y}{dx^7}$ is:

$$y = \alpha e^x + \beta e^{x\cos A\frac{2}{7}\pi}\sin A\left(x\sin A\frac{2}{7}\pi + \mathfrak{B}\right)$$
$$+ \gamma e^{x\cos A\frac{4}{7}\pi}\sin A\left(x\sin A\frac{4}{7}\pi + \mathfrak{C}\right)$$
$$+ \delta e^{x\cos A\frac{6}{7}\pi}\sin A\left(x\sin A\frac{6}{7}\pi + \mathfrak{D}\right).$$

8. The integral of this equation $0 = y - \frac{d^8y}{dx^8}$ is:

$$y = \alpha e^x + \beta e^{x\cos A\frac{1}{4}\pi}\sin A\left(x\sin A\frac{1}{4}\pi + \mathfrak{B}\right) + \gamma\sin A(x + \mathfrak{C})$$
$$+ \delta e^{x\cos A\frac{3}{4}\pi}\sin A\left(x\sin A\frac{3}{4}\pi + \mathfrak{D}\right) + \varepsilon e^{-x}.$$

etc.

Problem III

§40 If this differential equation of indefinite degree n was propounded

$$0 = y + \frac{d^ny}{dx^n},$$

having put the element dx constant, to find its integral.

Solution

Having written 1 instead of y and z^n instead of $\frac{d^ny}{dx^n}$, this algebraic equation will result $0 = 1 + z^n$, if which number n was odd, it will have the simple real divisor $1 + z$, from which $y = \alpha e^{-x}$ results. All remaining simple divisors are imaginary; but each two of them are contained in this real trinomial factor:

$$1 - 2z\cos A\frac{2k-1}{n}\pi + zz,$$

and this expression suggests all divisors of the form $1 + z^n$, if all odd numbers not larger than n are successively substituted for $2k - 1$. But having compared this formula

$$1 - 2z\cos A\frac{2k-1}{n}\pi + zz$$

to the general factor

$$ff - 2fz\cos A\varphi + zz,$$

we have

$$f = 1 \quad \text{and} \quad \varphi = \frac{2k-1}{n}\pi;$$

therefore, hence the following value for y arises

$$y = \alpha e^{x\cos A\frac{2k-1}{n}\pi}\sin A\left(x\sin A\frac{2k-1}{n}\pi + \mathfrak{A}\right).$$

And in this general value even the value of y which must originate from the simple factor $1+z$, if n was an odd number, is contained; for, in this case $y = \alpha e^{-x}$, if $2k-1 = n$; for, then

$$\cos A\frac{2k-1}{n}\pi = \cos A\pi = -1$$

and its sine $= 0$. Therefore, the complete integral of the propounded equation will be found, if in this form

$$y = \alpha e^{x\cos A\frac{2k-1}{n}\pi}\sin A\left(x\sin A\frac{2k-1}{n}\pi + \mathfrak{A}\right)$$

all odd numbers 1, 3, 5, 7 etc., which are not greater than the exponent n, are successively substituted for $2k-1$ and all those values are collected into one sum. Therefore, this way the complete integral in question will result

$$\begin{aligned}
y &= \alpha e^{x\cos A\frac{1}{n}\pi}\sin A\left(x\sin A\frac{1}{n}\pi + \mathfrak{A}\right)\\
&+ \beta e^{x\cos A\frac{3}{n}\pi}\sin A\left(x\sin A\frac{3}{n}\pi + \mathfrak{B}\right)\\
&+ \gamma e^{x\cos A\frac{5}{n}\pi}\sin A\left(x\sin A\frac{5}{n}\pi + \mathfrak{C}\right)\\
&+ \delta e^{x\cos A\frac{7}{n}\pi}\sin A\left(x\sin A\frac{7}{n}\pi + \mathfrak{D}\right)\\
&+ \text{etc},
\end{aligned}$$

which terms must be continued until n arbitrary constants will have entered; this will happen, if from the series

$$\frac{1}{n}, \quad \frac{3}{n}, \quad \frac{5}{n}, \quad \frac{7}{n}, \quad \text{etc}$$

everything not greater than 1 is taken. But the last term, if n is an even number, will be

$$\nu e^{x\cos A\frac{n-1}{n}\pi}\sin A\left(x\sin A\frac{n-1}{n}\pi+\mathfrak{N}\right).$$

But if n is an odd number, the last term will be:

$$\nu e^{-x},$$

the penultimate on the other hand

$$\mu e^{x\cos A\frac{n-2}{n}\pi}\sin A\left(x\sin A\frac{n-2}{n}\pi+\mathfrak{M}\right),$$

whence the complete integral is easily assigned in each case. Q.E.I.

§41 To explain this integral more diligently let us expand several simpler cases putting, as we did in the preceding problem, the integer numbers 1, 2, 3, $\cdots$, 8 for n; so that the ability of this integration is seen more clearly.

1. The integral of this equation $0=y+\frac{dy}{dx}$ is

$$y=\alpha e^{-x}.$$

2. The integral of this equation $0=y+\frac{ddy}{dx^2}$ is

$$y=\alpha\sin A(x+\mathfrak{A}).$$

3. The integral of this equation $0=y+\frac{d^3y}{dx^3}$ is

$$y=\alpha e^{x\cos A\frac{1}{3}\pi}\sin A\left(x\sin A\frac{1}{3}\pi+\mathfrak{A}\right)+\beta e^{-x}.$$

4. The integral of this equation $0=y+\frac{d^4y}{dx^4}$ is

$$y=\alpha e^{x\cos A\frac{1}{4}\pi}\sin A\left(x\sin A\frac{1}{4}\pi+\mathfrak{A}\right)$$
$$\beta e^{x\cos A\frac{3}{4}\pi}\sin A\left(x\sin A\frac{3}{4}\pi+\mathfrak{B}\right).$$

5. The integral of this equation $0 = y + \frac{\mathrm{d}^5 y}{\mathrm{d}x^5}$ is

$$y = \alpha \mathrm{e}^{x \cos A \frac{1}{5}\pi} \sin A \left(x \sin \frac{1}{5}\pi + \mathfrak{A} \right) + \beta \mathrm{e}^{x \cos A \frac{3}{5}\pi} \sin A \left(x \sin A \frac{3}{5}\pi + \mathfrak{B} \right) + \gamma \mathrm{e}^{-x}.$$

6. The integral of this equation $0 = y + \frac{\mathrm{d}^6 y}{\mathrm{d}x^6}$ is

$$y = \alpha \mathrm{e}^{x \cos A \frac{1}{6}\pi} \sin A \left(x \sin A \frac{1}{6}\pi + \mathfrak{B} \right) + \beta \sin A (x + \mathfrak{B}) + \gamma \mathrm{e}^{x \cos A \frac{5}{6}\pi} \sin A \left(x \sin A \frac{5}{6}\pi + \mathfrak{C} \right).$$

7. The integral of this equation $0 = y + \frac{\mathrm{d}^7 y}{\mathrm{d}x^7}$ is

$$y = \alpha \mathrm{e}^{x \cos A \frac{1}{7}\pi} \sin A \left(x \sin A \frac{1}{7}\pi + \mathfrak{A} \right) + \beta \mathrm{e}^{x \cos A \frac{3}{7}\pi} \sin A \left(x \sin A \frac{3}{7}\pi + \mathfrak{B} \right) + \gamma \mathrm{e}^{x \cos A \frac{5}{7}\pi} \sin A \left(x \sin A \frac{5}{7}\pi + \mathfrak{C} \right) + \delta \mathrm{e}^{-x}.$$

8. The integral of this equation $0 = y + \frac{\mathrm{d}^8 y}{\mathrm{d}x^8}$ is

$$y = \alpha \mathrm{e}^{x \cos A \frac{1}{8}\pi} \sin A \left(x \sin A \frac{1}{8}\pi + \mathfrak{A} \right) + \beta \mathrm{e}^{x \cos A \frac{3}{8}\pi} \sin A \left(x \sin A \frac{3}{8}\pi + \mathfrak{B} \right) + \gamma \mathrm{e}^{x \cos A \frac{5}{8}\pi} \sin A \left(x \sin A \frac{5}{8}\pi + \mathfrak{C} \right) + \delta \mathrm{e}^{x \cos A \frac{7}{8}\pi} \sin A \left(x \sin A \frac{7}{8}\pi + \mathfrak{D} \right).$$

Problem IV

§42 If this differential equation of order $2n$ was propounded:

$$0 = y + \frac{2h\mathrm{d}^n y}{\mathrm{d}x^n} + \frac{\mathrm{d}^{2n} y}{\mathrm{d}x^{2n}},$$

having put the element $\mathrm{d}x$ to be constant, to find its integral while $hh > 1$.

SOLUTION

If according to the rule we write 1 for y, z^n for $\frac{\mathrm{d}^n y}{\mathrm{d}x^n}$ and z^{2n} for $\frac{\mathrm{d}^{2n} y}{\mathrm{d}x^{2n}}$, this algebraic equation results

$$0 = 1 + 2hz^n + z^{2n},$$

which because of $hh > 1$ is resolved into these two factors:

$$\left[z^n + h + \sqrt{hh-1}\right]\left[z^n + h - \sqrt{hh-1}\right].$$

But we want to consider h as a positive quantity here; and for this reason, so

$$h + \sqrt{hh-1} \quad \text{as} \quad h - \sqrt{hh-1}$$

will be positive quantities. Therefore, let

$$h + \sqrt{hh-1} = a^n \quad \text{and} \quad h - \sqrt{hh-1} = b^n,$$

such that $ab = 1$. Therefore, this equation will be resolved into two factors:

$$0 = (z^n + a^n)(z^n + b^n)$$

and each real trinomial factor of the first will be contained in this form:

$$aa - 2az \cos A \frac{2n-1}{n}\pi + zz,$$

of the second on the other hand in this one:

$$bb - 2bz \cos A \frac{2k-1}{n}\pi + zz.$$

And one will have all factors, if in each of both forms successively all odd numbers 1, 3, 5, 7 etc. are written for $2k-1$

$$
\begin{aligned}
y = {} & A e^{ax\cos A\frac{1}{n}\pi}\sin A\left(ax\sin A\frac{1}{n}\pi+\mathfrak{A}\right)\\
& + B e^{ax\cos A\frac{3}{n}\pi}\sin A\left(ax\sin A\frac{3}{n}\pi+\mathfrak{B}\right)\\
& + C e^{ax\cos A\frac{5}{n}\pi}\sin A\left(ax\sin A\frac{5}{n}\pi+\mathfrak{C}\right)\\
& + D e^{ax\cos A\frac{7}{n}\pi}\sin A\left(ax\sin A\frac{7}{n}\pi+\mathfrak{D}\right)+\text{etc}\\
& + \alpha e^{bx\cos A\frac{1}{n}\pi}\sin A\left(bx\sin A\frac{1}{n}\pi+\mathfrak{a}\right)\\
& + \beta e^{bx\cos A\frac{3}{n}\pi}\sin A\left(bx\sin A\frac{3}{n}\pi+\mathfrak{b}\right)\\
& + \gamma e^{bx\cos A\frac{5}{n}\pi}\sin A\left(bx\sin A\frac{5}{n}\pi+\mathfrak{c}\right)+\text{etc}
\end{aligned}
$$

Q.E.I.

PROBLEM V

§43 If this differential equation of indefinite order $2n$ was propounded:

$$0 = y - \frac{2h\mathrm{d}^n y}{\mathrm{d}x^n} + \frac{\mathrm{d}^{2n} y}{\mathrm{d}x^{2n}},$$

having taken $\mathrm{d}x$ as a constant and while $hh > 1$, to find its integral.

SOLUTION

According to the rule given above here the following algebraic equation will result

$$0 = 1 - 2hz^n + z^{2n};$$

this is first resolved into these two real factors:

$$0 = \left[z^n - h + \sqrt{hh-1}\right]\left[z^n - h - \sqrt{hh-1}\right].$$

But since h denotes a positive quantity, put

$$h+\sqrt{hh-1}=a^n \quad \text{and} \quad h-\sqrt{hh-1}=b^n,$$

such that $ab=1$; and hence this equation will result

$$0=(z^n-a^n)(z^n-b^n).$$

All real trinomial factors of the first factor z^n-a^n are contained in this form

$$aa-2az\cos A\frac{2k}{n}\pi+zz,$$

of the second on the other hand z^n-b^n in this form

$$bb-2bz\cos A\frac{2k}{n}\pi+zz;$$

and one will have all factors, if in each of both forms for $2k$ all even numbers 0, 2, 4, 6 etc. not larger than n are successively substituted. Therefore, from these known factors the integral in question will be concluded to be:

$$y=\begin{cases} Ae^{ax} & +Be^{ax\cos A\frac{2}{n}\pi}\sin A\left(ax\sin A\frac{2}{n}\pi+\mathfrak{B}\right)\\ & +Ce^{ax\cos A\frac{4}{n}\pi}\sin A\left(ax\sin A\frac{4}{n}\pi+\mathfrak{C}\right)\\ & +De^{ax\cos A\frac{6}{n}\pi}\sin A\left(ax\sin A\frac{6}{n}\pi+\mathfrak{D}\right)+\text{etc}\\ +\alpha e^{bx} & +\beta e^{bx\cos A\frac{2}{n}\pi}\sin A\left(bx\sin A\frac{2}{n}\pi+\mathfrak{b}\right)\\ & +\gamma e^{bx\cos A\frac{4}{n}\pi}\sin A\left(bx\sin A\frac{4}{n}\pi+\mathfrak{c}\right)\\ & +\delta e^{bx\cos A\frac{6}{n}\pi}\sin A\left(bx\sin A\frac{6}{n}\pi+\mathfrak{d}\right)+\text{etc}\end{cases}$$

Q.E.I.

PROBLEM VI

§44 If this differential equation of indefinite order was propounded:

$$0=y+\frac{2h\mathrm{d}^ny}{\mathrm{d}x^n}-\frac{\mathrm{d}^{2n}y}{\mathrm{d}x^{2n}},$$

haven taken the element $\mathrm{d}x$ to be constant, to find its integral.

Solution

The algebraic equation resulting from this according to the prescriptions from this:

$$0 = 1 + 2hz^n - z^{2n},$$

is first resolved into these two real factors:

$$0 = \left[h + \sqrt{hh+1} - z^n\right]\left[-h + \sqrt{hh+1} + z^n\right].$$

Let, what because of the positive quantity h is always possible,

$$\sqrt{hh+1} + h = a^n \quad \text{and} \quad \sqrt{hh+1} - h = b^n,$$

such that $ab = 1$; and hence this equation will originate:

$$0 = (a^n - z^n)(b^n + z^n).$$

All real trinomial factors of the first factor $a^n - z^n$ are contained in this form

$$aa - 2az \cos A\frac{2k}{n}\pi + zz,$$

of the second on the other hand in this:

$$bb - 2bz \cos A\frac{2k-1}{n}\pi + zz,$$

and one will have all factors, if in the first for $2k$ all even numbers 0, 2, 4, 6 etc., in the second on the other hand for $2k-1$ all odd numbers 1, 3, 5, 7 etc. not exceeding n are substituted successively. Therefore, from the known factors the integral in question is concluded:

$$y = \begin{cases} Ae^{ax} & +Be^{ax\cos A\frac{2}{n}\pi}\sin A\left(ax\sin A\frac{2}{n}\pi + \mathfrak{B}\right) \\ & +Ce^{ax\cos A\frac{4}{n}\pi}\sin A\left(ax\sin A\frac{4}{n}\pi + \mathfrak{C}\right) \\ & +De^{ax\cos A\frac{6}{n}\pi}\sin A\left(ax\sin A\frac{6}{n}\pi + \mathfrak{D}\right) + \text{etc} \\ & +\alpha e^{bx\cos A\frac{1}{n}\pi}\sin A\left(bx\sin A\frac{1}{n}\pi + \mathfrak{a}\right) \\ & +\beta e^{bx\cos A\frac{3}{n}\pi}\sin A\left(bx\sin A\frac{3}{n}\pi + \mathfrak{b}\right) \\ & +\gamma e^{bx\cos A\frac{5}{n}\pi}\sin A\left(bx\sin A\frac{5}{n}\pi + \mathfrak{c}\right) + \text{etc} \end{cases}$$

Q.E.I.

Problem VII

§45 If this differential equation of indefinite order $2n$ was propounded

$$0 = y - \frac{2n\mathrm{d}^n y}{\mathrm{d}x^n} - \frac{\mathrm{d}^{2n} y}{\mathrm{d}x^{2n}},$$

in which the element dx is put constant, to find its integral.

Solution

By substitution do be done according to the rule given above hence this algebraic equation of order $2n$ results:

$$0 = 1 - 2hz^n - z^{2n},$$

which at first is split into these two real factors

$$0 = \left[-h + \sqrt{hh+1} - z^n\right]\left[h + \sqrt{hh+1} + z^n\right].$$

Because of the positive quantity h put

$$\sqrt{hh+1} + h = a^n \quad \text{and} \quad \sqrt{hh+1} - h = b^n,$$

such that $ab = 1$. And one will have the following equation which is to be resolved

$$0 = (a^n + z^n)(b^n - z^n),$$

all trinomial factors of the first factor of which, i.e. $a^n + z^n$, are contained in this form:

$$aa - 2az\cos A\frac{2k-1}{n}\pi + zz,$$

of the second on the other hand in this form

$$bb - 2bz\cos A\frac{2k}{n}\pi + zz;$$

and one will have all factors, if in that form successively all odd numbers 1, 3, 5, 7 etc. are written instead of $2k-1$, in the first on the other hand all even numbers 0, 2, 4, 6 etc. not exceeding n instead of k. And therefore, from these factors the complete integral in question will be calculated to be:

$$y = \begin{cases} & Ae^{ax\cos A\frac{1}{n}\pi}\sin A\left(ax\sin A\frac{1}{n}\pi + \mathfrak{A}\right) \\ & +Be^{ax\cos A\frac{3}{n}\pi}\sin A\left(ax\sin A\frac{3}{n}\pi + \mathfrak{B}\right) \\ & +Ce^{ax\cos A\frac{5}{n}\pi}\sin A\left(ax\sin A\frac{5}{n}\pi + \mathfrak{C}\right) + \text{etc} \\ +\alpha e^{bx} & +\beta e^{bx\cos A\frac{2}{n}\pi}\sin A\left(bx\sin A\frac{2}{n}\pi + \mathfrak{b}\right) \\ & +\gamma e^{bx\cos A\frac{4}{n}\pi}\sin A\left(bx\sin A\frac{4}{n}\pi + \mathfrak{c}\right) \\ & +\delta e^{bx\cos A\frac{6}{n}\pi}\sin A\left(bx\sin A\frac{6}{n}\pi + \mathfrak{d}\right) + \text{etc.} \end{cases}$$

Q.E.I.

Problem VIII

§46 If this equation of indefinite degree was propounded:

$$0 = y + \frac{2h\mathrm{d}^n y}{\mathrm{d}x^n} + \frac{\mathrm{d}^{2n} y}{\mathrm{d}x^{2n}},$$

having put the element dx to be constant and while $hh < 1$, to find its complete integral.

Solution

The algebraic equation of order $2n$ resulting from this is

$$0 = 1 + 2hz^n + z^{2n};$$

to find all its real trinomial factors in the circle whose radius is $= 1$ take the arc ω the cosine of which is $= h$ such that $h = \cos A\omega$. Having found this arc each trinomial factor will be contained in this form:

$$1 - 2z\cos A\frac{k\pi - \omega}{n} + zz,$$

by substituting all odd numbers $1, 3, 5, 7, \cdots, 2n-1$ instead of k such that the number of these factors will be n, as the number of dimensions requires it. Therefore, from these known factors according to the given prescriptions the integral in question of the propounded equation will be found:

$$y = \begin{cases} & \alpha e^{x \cos A \frac{\pi - \omega}{n}} \sin A \left(x \sin A \frac{\pi - \omega}{n} + \mathfrak{a} \right) \\ + & \beta e^{x \cos A \frac{3\pi - \omega}{n}} \sin A \left(x \sin A \frac{3\pi - \omega}{n} + \mathfrak{b} \right) \\ + & \gamma e^{x \cos A \frac{5\pi - \omega}{n}} \sin A \left(x \sin A \frac{5\pi - \omega}{n} + \mathfrak{c} \right) \\ + & \text{etc} \\ + & \nu e^{x \cos A \frac{(2n-1)\pi - \omega}{n}} \sin A \left(x \sin A \frac{(2n-1)\pi - \omega}{n} + \mathfrak{n} \right). \end{cases}$$

Of course, the number of terms constituting this integral is n, and hence the number of entering arbitrary constants is $2n$, as the order of differentials of the propounded equation requires.

Q.E.I.

PROBLEM IX

§47 While again $hh < 1$, if this differential equation of indefinite degree $2n$ was propounded:

$$0 = y - \frac{2h d^n y}{dx^n} + \frac{d^{2n} y}{dx^{2n}},$$

having put the element dx to be constant, to find its complete integral.

SOLUTION

The algebraic equation which is deduced from this according to the given prescriptions is

$$0 = 1 - 2hz^n + z^{2n};$$

each real trinomial factor of it, the total amount of which is n, is contained in this general form:

$$1 - 2z \cos A \frac{k\pi - \omega}{n} + zz,$$

if for $2k$ successively all even numbers 2, 4, 6, 8 etc. to $2n$ inclusively are substituted. But here as before ω denotes the arc of the circle the cosine of which is h, which because of $h < 1$ can always be assigned such that $h = \cos A\omega$. But having known all factors of the equation

$$0 = 1 - 2hz^n + z^{2n},$$

the complete integral of the propounded differential equation will be:

$$y = \begin{cases} & \alpha e^{x\cos A\frac{2\pi-\omega}{n}} \sin A\left(x\sin A\frac{2\pi-\omega}{n} + \mathfrak{a}\right) \\ + & \beta e^{x\cos A\frac{4\pi-\omega}{n}} \sin A\left(x\sin A\frac{4\pi-\omega}{n} + \mathfrak{b}\right) \\ + & \gamma e^{x\cos A\frac{6\pi-\omega}{n}} \sin A\left(x\sin A\frac{6\pi-\omega}{n} + \mathfrak{c}\right) \\ + & \delta e^{x\cos A\frac{8\pi-\omega}{n}} \sin A\left(x\sin A\frac{8\pi-\omega}{n} + \mathfrak{d}\right) \\ + & \text{etc} \\ + & \nu e^{x\cos A\frac{2n\pi-\omega}{n}} \sin A\left(x\sin A\frac{2n\pi-\omega}{n} + \mathfrak{n}\right) \end{cases}$$

For, $2n$ arbitrary constants enter into this expression. Q.E.I.

Problem X

§48 If this differential equation of indefinite order was propounded:

$$0 = y \pm \frac{2\mathrm{d}^n y}{\mathrm{d}x^n} + \frac{\mathrm{d}^{2n} y}{\mathrm{d}x^{2n}},$$

in which the differential $\mathrm{d}x$ was put constant, to find its integral.

Solution

The algebraic equation which is hence formed reads:

$$0 = 1 \pm 2z^n + z^{2n} = (1 \pm z^n)^2;$$

since it is a square all its factors will be squares; therefore, for the upper sign this form

$$(1 - 2z\cos A\frac{2k-1}{n}\pi + zz)^2$$

contains all factors; for the lower this form

$$(1 - 2z\cos A\frac{2k}{n}\pi + zz)^2.$$

From these known factors for the lower signs of the equation

$$0 = y - \frac{2\mathrm{d}^{\mathrm{n}} y}{\mathrm{d}x^n} + \frac{\mathrm{d}^{2\mathrm{n}} y}{\mathrm{d}x^{2n}}$$

one will find the complete integral:

$$y = \begin{cases} Ae^x & +Be^{x\cos A\frac{2}{n}\pi}\sin A\left(x\sin A\frac{2}{n}\pi+\mathfrak{B}\right) \\ & +Ce^{x\cos A\frac{4}{n}\pi}\sin A\left(x\sin A\frac{4}{n}\pi+\mathfrak{C}\right) \\ & +\text{etc} \\ +\alpha xe^x & +\beta xe^{x\cos A\frac{2}{n}\pi}\sin A\left(x\sin A\frac{2}{n}\pi+\mathfrak{b}\right) \\ & +\gamma xe^{x\cos A\frac{4}{n}\pi}\sin A\left(x\sin A\frac{4}{n}\pi+\mathfrak{c}\right) \\ & +\text{etc.} \end{cases}$$

But the integral of the equation

$$0 = y + \frac{2\mathrm{d}^n y}{\mathrm{d}x^n} + \frac{\mathrm{d}^{2n}y}{\mathrm{d}x^{2n}}$$

will be

$$y = \begin{cases} & Ae^{x\cos A\frac{1}{n}\pi}\sin A\left(x\sin A\frac{1}{n}\pi+\mathfrak{A}\right) \\ + & Be^{x\cos A\frac{3}{n}\pi}\sin A\left(x\sin A\frac{3}{n}\pi+\mathfrak{B}\right) \\ + & Ce^{x\cos A\frac{5}{n}\pi}\sin A\left(x\sin A\frac{5}{n}\pi+\mathfrak{C}\right) \\ + & \text{etc} \\ + & \alpha xe^{x\cos A\frac{1}{n}\pi}\sin A\left(x\sin A\frac{1}{n}\pi+\mathfrak{a}\right) \\ + & \beta xe^{x\cos A\frac{3}{n}\pi}\sin A\left(x\sin A\frac{3}{n}\pi+\mathfrak{b}\right) \\ + & \gamma xe^{x\cos A\frac{5}{n}\pi}\sin A\left(x\sin A\frac{5}{n}\pi+\mathfrak{c}\right) \\ + & \text{etc} \end{cases}$$

Q.E.I.

§49 From these mentioned examples it is now seen abundantly clearly how all differential equations of each other contained in this form

$$0 = Ay + \frac{B\mathrm{d}y}{\mathrm{d}x} + \frac{C\mathrm{dd}y}{\mathrm{d}x^2} + \frac{D\mathrm{d}^3y}{\mathrm{d}x^3} + \frac{E\mathrm{d}^4y}{\mathrm{d}x^4} + \text{etc},$$

while the letters A, B, C, D etc. denote arbitrary constants must be treated and their complete integral must be found. For, the only difficulties resides in the resolution of algebraic equations into real either simpler or trinomial factors; but in this task we can justly assume this resolution, just depending on algebra, to be given. But this same method can indeed also be used in equations of this kind the terms of which continue to infinity, as long as all roots of the algebraic equations which are formed from this can be assigned. Therefore, we will illustrate the use in one example, and I will explain the integration of differential equations of infinite order in more detail on another occasion.

PROBLEM XI

§50 If this differential equation running to infinity was propounded:

$$0 = y - \frac{ddy}{2dx^2} + \frac{d^4y}{24dx^4} - \frac{d^6y}{720dx^6} + \frac{d^8y}{40320dx^8} - \text{etc},$$

in which the differential dx is put constant, to find its complete integral.

SOLUTION

Having written 1 for y and z^k for the differential of the corresponding order $\frac{d^ky}{dx^k}$, this equation running to infinity will arise

$$0 = 1 - \frac{z^2}{1 \cdot 2} + \frac{z^4}{1 \cdot 2 \cdot 3 \cdot 4} - \frac{z^6}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} + \frac{z^8}{1 \cdot 2 \cdots 8} - \text{etc},$$

which agrees with this one

$$0 = \cos Az.$$

Therefore, all roots of this equation are arcs of the circle with radius $= 1$ the cosine of which vanishes. Therefore, all possible value of z will be the following:

$$\pm\frac{\pi}{2}, \quad \pm\frac{3}{2}\pi, \quad \pm\frac{5}{2}\pi, \quad \pm\frac{7}{2}\pi, \quad \pm\frac{9}{2}\pi, \quad \text{etc.}$$

Therefore, having known these roots and hence all simple divisors of that equation, which are all real, the complete integral of the propounded differential equation will be

$$y = \alpha e^{\frac{\pi x}{2}} + \mathfrak{a}e^{-\frac{\pi x}{2}} + \beta e^{\frac{3\pi x}{2}} + \mathfrak{b}e^{-\frac{3\pi x}{2}} + \gamma e^{\frac{5\pi x}{2}} + \mathfrak{c}e^{-\frac{5\pi x}{2}} + \delta e^{\frac{7\pi x}{2}} + \mathfrak{d}e^{-\frac{7\pi x}{2}} + \dots$$

(to infinity)

And each term taken separately or several combined will give a particular integral of the propounded differential equation.

Q.E.I.